Hall Current and Inlet Disturbances in Constant Area Channel Flows

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The problem considered is that of obtaining the fluid flow fields introduced within an electromagnetic accelerator, by the off-axis component of the Lorentz force created when a "Hall component" of the electric current exists. In addition, the growth or decay of inlet disturbances is investigated. In this paper a perturbation solution for the fluid properties is obtained. The reference flow about which the perturbation is taken is a modified channel-flow solution which becomes exact as the ratio of channel height to channel length approaches zero. Several example solutions are computed, all but one of which lead to closed form expressions for the dependent variables in terms of the velocity of the imposed channel flow. The solutions indicate that for values of the Hall parameter near or greater than unity, the validity of the channel-flow solutions will be questioned in some cases of high electromagnetic loading, unless the channel width is kept very small. The results for the behavior of inlet disturbances show that in all cases considered such disturbances grow to some extent downstream.

Nomenclature

ρ	= density
P	= pressure
R	= gas constant
T	= temperature
S	= entropy
$\mathbf{u}$	= vector velocity
$\mathbf{w}$	= vector vorticity
$\mathbf{E}$	= vector electric field
$\mathbf{B}$	= vector magnetic field
$\mathbf{j}$	= vector electric current
σ	= electric conductivity
n_e	= electron density
e	= charge on electron
$\omega\tau$	= Hall parameter = $\sigma B/en_e$
μ_0	= magnetic permeability
x,y,z	= Cartesian coordinates
x'	= contracted axial coordinate = $x/[1 + (\omega\tau)^2]$
E	= y component of electric field (used in section on channel-flow analysis)
B	= z component of magnetic field (used in section on channel-flow analysis)
J	= pseudo current = $\sigma(E - uB)$
u	= axial component of fluid velocity
v	= y component of fluid velocity
w	= z component of fluid velocity
M	= Mach number
ξ	= dimensionless axial coordinate = $x(\sigma B_0{}^2/\bar{\rho}\bar{u})/[1 + (\omega\tau)^2]$
η	= dimensionless y coordinate = $y(\sigma B_0{}^2/\bar{\rho}\bar{u})/[1 + (\omega\tau)^2]$
$\bar{u},\bar{\rho},\bar{P}$, etc.	= channel-flow velocity, density, etc.
$\bar{u}_0,\bar{\rho}_0$, etc.	= channel-flow velocity, density, etc., at inlet to channel
u',ρ',P', etc.	= property perturbations about channel-flow properties
μ	= dimensionless channel-flow velocity = $\bar{u}/\bar{u}_0$
j	= dimensionless current $J/\sigma E_0$
χ	= dimensionless magnetic field = B/B_0
h	= width of channel (z direction)
H	= height of channel (y direction)
L	= length of channel (x direction)
R_M	= magnetic Reynold's number based on machine length = $\sigma\mu_0\bar{u}_0L$
Y	= dimensionless slope of axial velocity perturbation = $(\partial u'/\bar{u})/\partial\eta$

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Introduction

THE concept of steady-state electromagnetic acceleration or deceleration of a gas is now a familiar one, and many experimental and theoretical programs are under way to explain and predict the behavior of gases under the influence of electric and magnetic fields. Sears and Resler,[1] in an early paper, introduced the concept of a channel-flow analysis adapted to the description of gas flows with electromagnetic effects included. Since the introduction of the channel-flow concept, many special cases of channel flows have been considered (see for example Ref. 2), and the approximate results, obtained so easily by a channel-flow analysis, have led to easy interpretation of the average effects on the gas stream of the electromagnetic energy input.

There are several effects which act to limit the range of validity of channel-flow solutions, however, and the effects of these limitations have been the subject of considerable controversy in the past years. Perhaps the greatest controversy has centered about the effects of ignoring Maxwell's (fourth) equation for the curl of the magnetic field, when obtaining a channel-flow solution. McCune and Sears[3] defend the viewpoint that it is necessary neither to require the "magnetic Reynolds number" to equal zero (i.e., in effect to say that any magnetic field component generated by the current within the gas is inconsequential because such a field is very small compared to the applied field), nor to satisfy Maxwell's fourth equation exactly in the strictly one-dimensional case by requiring the axial rate of change of the applied magnetic field to be equated to (minus) the magnetic permeability of the gas times the applied current density.[4] By removing this latter restriction the magnetic field can be applied independently of the current, and a family of possible solutions exists where a single possible solution would exist if the magnetic field and current were coupled.

Following the reasoning of McCune and Sears, one then sees that, in fact, the magnetic field lines would curve across the width of the channel (Fig. 1) so that truly one-dimensional flow would not exist. That is, the off-axis Lorentz forces created by the induced axial component of the magnetic field would themselves induce off-axis velocities. It is shown in Refs. 3 and 5 that such velocities are extremely small compared to the average axial velocity and, in fact, the effect of Maxwell's fourth equation is experienced to the order of the width of the channel divided by the length of the channel, only through the curving of the magnetic field lines.

A second effect of importance that affects the possible range of validity of the channel-flow solutions is that brought

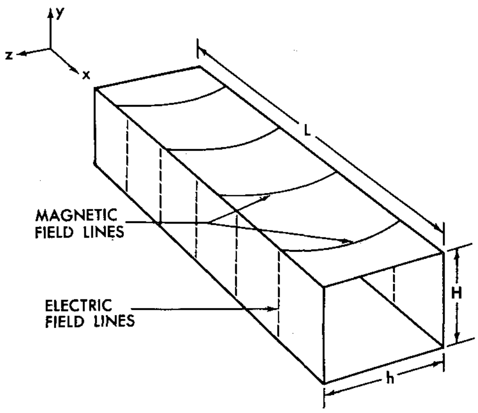

Fig. 1 Coordinate system and field lines.

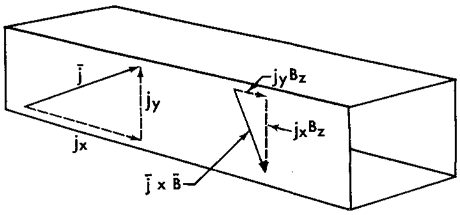

Fig. 2 Current components and Lorentz force components.

about by the off-axis component of the body force existing when there is an axial component (or "Hall component") of the applied current (Fig. 2). It is shown in Ref. 2 that within the framework of the channel-flow solutions, the effect of the change in the axial Lorentz force brought about by the existence of a Hall current is easily accounted for by a simple transformation of the axial coordinate.

This papers considers the effect of the Hall component of the current on the two-dimensional flow field, of a nonviscous compressible fluid, and investigates the perturbations on purely axial flow that are introduced. In the initial section of the paper, the equations are solved for the case where the Hall current is allowed to flow unimpeded, such as would be the case for a continuous electrode machine. The perturbations in the fluid parameters resulting from such a geometry will represent an upper limit to those to be found in actual machines. An equation for the perturbations in fluid properties is developed which gives the properties in terms of the velocity of the imposed channel flow solution. Four examples are considered, three of which lead to closed form solutions. These closed form solutions are particularly suitable for interpretive purposes, and in fact show that in some cases of high electromagnetic loading, the validity of the channel-flow solutions will be quite restricted.

In the final section of the paper, the equations are solved for zero Hall current, but for an assumed asymmetric inlet condition. This is the situation which could conceivably arise in an accelerator or generator with segmented electrodes, when, for some reason, the inlet flow is perturbed. It is found that in all cases investigated such inlet perturbations grow to some extent within the machine.

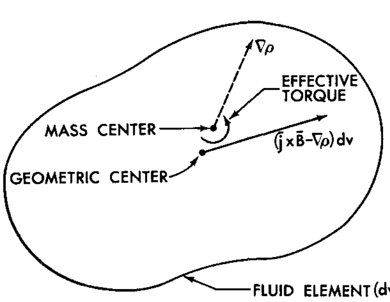

Fig. 3 Torque existing on fluid element when density gradient is present.

Vector Form of the Equations

The usual steady-state form of the equations of magnetogasdynamics will be repeated here, with the slight modification that the energy equation and equation of state will be written in terms of the entropy rather than the temperature. Assuming the fluid to be a perfect gas, and assuming that the electric current arises from the flow of electrons only, one then obtains

Continuity

$$\nabla \cdot \rho \mathbf{u} = 0 \tag{1}$$

Momentum

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \mathbf{j} \times \mathbf{B} \tag{2}$$

Energy

$$P(\mathbf{u}\cdot\nabla)S/R = \mathbf{j}^2/\sigma \tag{3}$$

State

$$S - S_0 = C_v \ln(P/P_0) - C_P \ln(\rho/\rho_0) \tag{4}$$

Ohm's Law

$$\mathbf{j} = \sigma\left\{\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e}\mathbf{j} \times \mathbf{B}\right\} \tag{5}$$

Second Maxwell Equation

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

Third Maxwell Equation

$$\nabla \times \mathbf{E} = 0 \tag{7}$$

Fourth Maxwell Equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{8}$$

In addition, one may write the current continuity equation which follows from Eq. (8), i.e.,

$$\nabla \cdot \mathbf{j} = 0 \tag{9}$$

Shercliff[6] and Sutton[7] have considered the behavior of an incompressible, electrically conducting fluid at entry to a region of electric and magnetic fields. In their work the importance of describing and interpreting the vorticity behavior was evident. In what follows, an equation will be developed that gives the vorticity behavior of a compressible fluid in a form suitable for interpretation.

The momentum equation may be written

$$\nabla(\mathbf{u}^2/2) + \boldsymbol{\omega} \times \mathbf{u} + (1/\rho)\nabla P = (1/\rho)\mathbf{j} \times \mathbf{B}$$

Taking the curl of this equation, and making use of the continuity equation and the result that the divergences of the current, magnetic field, and vorticity are all zero leads to

$$\rho(\mathbf{u}\cdot\nabla)\frac{\boldsymbol{\omega}}{\rho} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nabla\frac{1}{\rho} \times [\mathbf{j} \times \mathbf{B} - \nabla P] + 1/\rho[(\mathbf{B}\cdot\nabla)\mathbf{j} - (\mathbf{j}\cdot\nabla)\mathbf{B}]$$

This form of the equation allows easy interpretation of the various effects, as we see that the equating of the first term on either side of the equation represents the tendency of the fluid to conserve circulation if no rotating influences are present.[8]

The second term on the right side of the equation represents the contribution of compressibility to the tendency to generate vorticity. That is, if the gradient of the density is zero (or parallel to the vector $\mathbf{j} \times \mathbf{B} - \nabla P$) this term does not contribute. Interpretation in this case is aided by Fig. 3, in which one sees that the net vector volume force $(\mathbf{j} \times \mathbf{B} - \nabla P)dV$ acts through the geometric center of the fluid element. On the other hand, if a density gradient is present, the mass centroid will not be coincident with the geometric center of the element, and a torque about the mass center will exist, which in turn will lead to fluid rotation.

Finally, the last term on the right of the equation can be seen, with the aid of Fig. 4, to represent the torque introduced on the element if the Lorentz force varies in either the magnetic field or current direction. It can be seen that a change in the Lorentz force in a plane perpendicular to the plane containing the current and magnetic field will not contribute to the rotation of the element, because such a change would be in the direction of the Lorentz force itself, and hence would not affect the torque.

Thus, one sees that the last two terms represent, respectively, the compressible and incompressible effects of the electromagnetic loading on the vorticity. It will be evident in what follows that the effect of compressibility will often dominate over the incompressible effects, and it is for this reason that it is most unsuitable to attempt to generalize the results of incompressible flow theory to compressible flow.

Channel Flow Equations

The set of equations, Eqs. (1–9), may be written in the form of channel flow equations (i.e., in a form in which all properties are considered to be constant across the flow cross section) by simply including only the terms that give the axial variation of properties. The physical reality, or lack of reality, of the solutions given by the resultant set of equations will depend largely on the magnetic and electric field loadings of the device considered, as well as the range of gas properties found in the device. In a later section the extent to which the channel-flow solutions will be in error will be discussed, but for now the channel-flow equations will be listed for convenience.

Throughout the work considered in this paper, both the conductivity and Hall parameter will be considered constant, an assumption not greatly at variance with the facts in the range of operation usually encountered in the "freestream" region of present day machines. With this assumption one obtains from Eqs. (1–5), for constant area flow

Continuity

$$(d/dx')(\rho u) = 0 \qquad (10)$$

Momentum

$$\rho u(du/dx') + (dP/dx') = JB \qquad (11)$$

Energy

$$Pu(d/dx')(S/R) = J^2/\sigma \qquad (12)$$

State

$$S - S_0 = C_v \ln(P/P_0) - C_P \ln(\rho/\rho_0) \qquad (13)$$

Ohm's Law

$$J = \sigma(E - uB) \qquad (14)$$

This group of equations is exactly that which is usually considered in the analysis of channel flows, and a great many solution (for various prescribed field loadings) have been obtained. In writing the equations in this form the effects of the finite Hall parameter have been accounted for by transforming the axial coordinate x to the new coordinate $x' = x/[1 + (\omega\tau)^2]$, and in addition, the "pseudo current" $J = \sigma(E - uB)$ has been defined which is in fact just $[1 + (\omega\tau)^2]$ times the y component of the electric current. It is obvious that this form

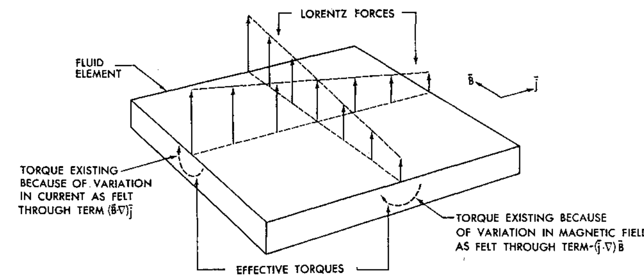

Fig. 4 Torque existing on fluid element when variation in Lorentz force is present.

of the equations becomes identical to the normal channel-flow equations as the Hall parameter approaches zero.

Solutions to the Channel Flow Equations

It will be of use, in the following sections, to have the results of several channel-flow solutions available, and for this reason two special forms of the channel-flow solution given in Ref. 1 will be developed. In addition, examples from Refs. 2 and 9 will be given.

In Ref. 1, the differential equations of constant area channel flow were manipulated to give, in the notation of Eqs. (10–14),

$$\frac{du}{dx'} = \frac{\gamma M^2}{M^2 - 1}\left(\frac{\sigma B^2}{\rho u}\right)\frac{1}{u}\left(\frac{E}{B} - u\right)\left(u - \frac{\gamma - 1}{\gamma}\frac{E}{B}\right) \qquad (15)$$

$$\frac{dM}{dx'} = \frac{1 + [(\gamma - 1)/2]M^2}{M^2 - 1}\left(\frac{\sigma B^2}{\rho u}\right)\frac{\gamma M^3}{u^2}\left(\frac{E}{B} - U\right) \times \left(u - \frac{1 + \gamma M^2}{1 + [(\gamma - 1)/2]M^2}\,\frac{\gamma - 1}{2\gamma}\,\frac{E}{B}\right) \qquad (16)$$

Now, Sears and Resler considered the situation where the electric and magnetic field were varied such that their ratio was proportional to the fluid velocity. The constant of proportionality was chosen so that the acceleration, at a prescribed Mach number, would be a maximum, which leads to the relationship

$$\frac{E}{uB} = \frac{2\gamma - 1}{2(\gamma - 1)} \qquad (17)$$

With this expression substituted into Eq. (16), it can be seen that the axial rate of change of Mach number is zero if the Mach number itself is given by

$$M = \{(2\gamma + 1)/\gamma\}^{1/2} \qquad (18)$$

Thus, restricting our attention to this special case, we find that Eq. (15) becomes in terms of the dimensionless variables χ, μ, and ξ

$$\frac{d\mu}{d\xi} = \frac{2\gamma + 1}{4(\gamma^2 - 1)}\,\mu\chi^2 \qquad (19)$$

With the aid of the dimensionless form of Eq. (17), Eq. (19) may be easily integrated for the two special cases: 1) constant applied electric field, and 2) constant applied magnetic field. With the results so obtained, and with the use of Eqs. (12) and (14), the relationships given in Table 1 follow easily.

In a similar way, simple special cases of the results of Kerrebrock and Oates (Refs. 2, 9, and 10) may be written in the appropriate form so that the results summarized in Table 1 are obtained.

Perturbation Variables and Order of Magnitudes

In the following sections, solutions will be obtained using perturbation techniques in which the reference flow about

Table 1 Summary of results of channel flow solutions

Case	Constraints	Mach number	Dimensionless current, $j = \dfrac{\sigma(E - uB)}{\sigma E_0} = \dfrac{J}{\sigma E_0}$	Dimensionless magnetic field, $\chi = \dfrac{B}{B_0}$	Dimensionless entropy variation, $\dfrac{d\bar{S}/C_p}{d\mu}$
1)	$\dfrac{E}{uB} = \dfrac{2\gamma - 1}{2(\gamma - 1)}$, $E =$ Const	$M^2 = \dfrac{2\gamma + 1}{\gamma}$	$\dfrac{1}{2\gamma - 1}$	$\dfrac{1}{\mu}$	$\dfrac{\gamma + 1}{\gamma}\dfrac{1}{\mu}$
2)	$\dfrac{E}{uB} = \dfrac{2\gamma - 1}{2(\gamma - 1)}$, $B =$ Const	$M^2 = \dfrac{2\gamma + 1}{\gamma}$	$\left(\dfrac{1}{2\gamma - 1}\right)\mu$	1	$\dfrac{\gamma + 1}{\gamma}\dfrac{1}{\mu}$
3)	$T =$ Const, $E =$ Const	$M = \mu$	$\dfrac{1}{\gamma}\dfrac{1}{\mu^2}$	$\dfrac{\gamma}{\gamma - 1}\dfrac{1}{\mu}\left(1 - \dfrac{1}{\gamma\mu^2}\right)$	$\dfrac{\gamma - 1}{\gamma}\dfrac{1}{\mu}$
4)	$E =$ Const, $B =$ Const	$M^2 = \dfrac{2}{\dfrac{\gamma + 1}{\mu} - (\gamma - 1)}$	$1 - \dfrac{\gamma - 1}{\gamma}\mu$	1	$\dfrac{\gamma+1}{\gamma\mu}\dfrac{\gamma - (\gamma - 1)\mu}{(\gamma + 1) - (\gamma - 1)\mu}$

which the perturbation variables are taken is considered to be that flow given by solution of the modified channel-flow equations, Eqs. (10–14). The axial behavior of the fluid properties so obtained may, in fact, be considered to be the behavior of the average fluid properties. Denoting these average properties by ρ, $\bar{u}$, etc., we then define the perturbation variables by the relationships $\rho = \bar{\rho} + \rho'$, $u = \bar{u} + u'$, $v = v'$, $w = w'$, etc.

The order of magnitudes of the terms $\rho'/\bar{\rho}$, $u'/\bar{u}$, $v'/\bar{u}$, etc., are of great interest, and for this reason the gross effects of the finite Hall parameter on these fluid properties will be considered. As mentioned in the introduction, the effect of Maxwell's fourth equation is experienced to first order in h/L only in the curving of the magnetic field lines, the effect on the velocity components being of second order.[3,5]

It is easily shown from Eq. (5) that when the fluid velocity of a channel is predominately axial, then the axial component of the electric current is related to the y component of the electric current (Fig. 2) by the equation,

$$j_x \approx -\omega\tau j_y$$

In this case the pressure gradient across the channel follows from the y component of the momentum equation to give

$$\frac{\partial P}{\partial y} \approx -j_x B_z \approx \omega\tau j_y B_z$$

Integration of this expression then gives for the pressure change across the channel $\Delta P_y \approx \omega\tau j_y B_z H$. The magnitude of this change in pressure may thus be compared to the pressure itself which may be estimated from the axial momentum equation to be at least as large as $P_0 \approx j_y B_z L$. The ratio of the pressure change to the reference pressure is thus seen to be

$$\Delta P_y/P_0 \approx \omega\tau H/L$$

It is thus apparent that if the Hall parameter is of order unity, the changes in fluid properties will be of first order in H/L. In the following sections analytical expressions for these first order perturbations will be obtained.

The effect of the off-axis Lorentz force on the y component of velocity is of interest, and one sees that, using the result that Eq. (10) is satisfied by the channel-flow quantities $\bar{\rho}$ and $\bar{u}$, the continuity equation gives to first order

$$\frac{\partial}{\partial_x}\left(\frac{\rho'}{\bar{\rho}} + \frac{u'}{\bar{u}}\right) + \frac{\partial}{\partial y}\left(\frac{v'}{\bar{u}}\right) = 0$$

Thus, one may assume that the cross-channel variation of the perturbation y velocity is of the same order as the axial variation of the perturbation axial velocity. Now, because the perturbation in axial velocity will be allowed to build up over the entire length of the machine, one may estimate the magnitude of the axial derivative by assuming $\partial(u'/\bar{u})/\partial x \approx (u'/\bar{u})(1/L)$. The y velocity, however, is constrained to have a value of zero at the walls (by the assumption of constant-area flow), so one sees that integration of the continuity equation in the y direction gives the result

$$v'/\bar{u} \approx (u'/\bar{u})(H/L) = O(H/L)^2$$

This important result shows that the effects of the y component of velocity will not be experienced to first order in the geometry considered in this paper, and for this reason the continuity equation need not be considered in the analysis˙ This result is also reflected in the perturbation forms of the remaining equations in that the y component of velocity dose not enter. In particular, the y component of the momentum equation then states that the Lorentz force is completely balanced by the pressure gradient, the y component of the inertial effects being of smaller order.

As a final consideration in this section we see from Eq. (7), using similar arguments to those just given, that the axial component of the electric field for this case of zero axial electric field along the center of the channel, is given by $E_x \approx E_y(H/L)$.

Thus, though the terms $\partial E_x/\partial y$ and $\partial E_y/\partial x$ are equal, the axial electric field component is of smaller order than the y component of electric field.

Perturbation Equations

With the result of the last section, that to the first order in H/L we may exclude both the effects of Maxwell's fourth equation and the y component of velocity, Eqs. (2–9) may be written in component form to give

Momentum, x

$$\rho u\frac{\partial u}{\partial x'} + \frac{\partial P}{\partial x'} = \sigma B_0^2\left[\left(\frac{E_y}{B_0} - u\chi\right) + \omega\tau\frac{E_x}{B_0}\right]\chi \quad (20)$$

Momentum, y

$$\partial P/\partial y' = \sigma B_0^2\left[\omega\tau\left(\frac{E_y}{B_0} - u\chi\right) - \frac{E_x}{B_0}\right]\chi \quad (21)$$

Energy

$$Pu\frac{\partial S/R}{\partial x'} = \sigma B_0^2\left(\frac{E_y}{B_0} - u\chi\right)^2 \quad (22)$$

Current Divergence

$$\frac{\partial}{\partial y'}\left[\left(\frac{E_y}{B_0} - u\chi\right) + \omega\tau\frac{E_x}{B_0}\right] = \frac{\partial}{\partial x'}\left[\omega\tau\left(\frac{E_y}{B_0} - u\chi\right) - \frac{Ex}{B_0}\right] \quad (23)$$

3rd Maxwell Eq.

$$\frac{\partial E_y}{\partial x'} = \frac{\partial E_x}{\partial y'} \quad (24)$$

Taking the derivative with respect to y' of Eq. (20), subtracting the derivative with respect to x' of Eq. (21), and using Eqs. (23) and (24) one obtains

$$\frac{\partial \rho u}{\partial y'}\left(\frac{\partial u}{\partial x'}\right) + \rho u \frac{\partial}{\partial x'}\left(\frac{\partial u}{\partial y'}\right) = -\sigma B_0^2\left[\omega\tau\left(\frac{E_y}{B_0} - u\chi\right) - \frac{E_x}{B_0}\right]\frac{\partial \chi}{\partial x'} \quad (25)$$

The derivative with respect to y' of Eq. (22) gives, with Eqs. (23) and (24)

$$\frac{\partial P u}{\partial y'}\frac{\partial S/R}{\partial x'} + Pu\frac{\partial}{\partial x'}\left(\frac{\partial S/R}{\partial y'}\right) = -2\sigma B_0^2\left(\frac{E_y}{B_0} - u\chi\right)\frac{\partial}{\partial x'}\left(\omega\tau u\chi + \frac{E_x}{B_0}\right) \quad (26)$$

The equation of state, Eq. (4), gives to first order

$$S - \bar{S} = S' = C_v P'/\bar{P} - C_P \rho'/\bar{\rho} \quad (27)$$

Thus, to first order substitution of the perturbation variables into Eqs. (25) and (26) gives with the aid of Eq. (21),

$$\left\{\frac{2\partial u'/\bar{u}}{\partial y'} + \frac{\partial \rho'/\bar{\rho}}{\partial y'}\right\}\frac{\partial \bar{u}}{\partial x'} + \bar{u}\frac{\partial}{\partial x'}\left(\frac{\partial u'/\bar{u}}{\partial y'}\right) = -\omega\tau\left(\frac{\sigma B_0^2}{\bar{\rho}\bar{u}}\right)\left(\frac{\bar{E}_y}{B_0} - \bar{u}\chi\right)\frac{\partial \chi}{\partial x'} \quad (28)$$

$$\frac{\partial u'/\bar{u}}{\partial y'}\frac{\partial \bar{S}/R}{\partial x'} - \frac{C_P}{R}\frac{\partial}{\partial x'}\left(\frac{\partial \rho'/\bar{\rho}}{\partial y'}\right) = -\sigma B_0^2\omega\tau\left[\left(\frac{{}_y\bar{E}}{B_0} - \bar{u}\chi\right)\left(\frac{2}{\bar{\rho}\bar{u}}\frac{\partial \bar{u}\chi}{\partial x'} + \frac{\chi}{\bar{\rho}}\frac{\partial \bar{S}/R}{\partial x'}\right) + \frac{C_v}{R}\frac{\partial}{\partial x'}\left\{\frac{\chi}{\bar{\rho}}\left(\frac{{}_y\bar{E}}{B_0} - \bar{u}\chi\right)\right\}\right] \quad (29)$$

These two equations allow solution for the two unknowns $\partial(u'/\bar{u})/\partial y'$ and $\partial(\rho'/\bar{\rho})/\partial y'$ in terms of the imposed channel-flow variables. A far more convenient form of these two equations is obtained in terms of the dimensionless variables μ, j, η, etc.

It is also convenient to replace the independent variable x' by the dimensionless channel-flow velocity to obtain

$$\frac{\partial \rho'/\bar{\rho}}{\partial \eta} + 2\frac{\partial u'/\bar{u}}{\partial \eta} + \mu\frac{\partial}{\partial \mu}\left(\frac{\partial u'/\bar{u}}{\partial \eta}\right) = -\omega\tau\frac{E_{y0}/B_0}{u_0}\frac{d\chi}{d\mu} \quad (30)$$

$$\frac{\partial}{\partial \mu}\left(\frac{\partial \rho'/\bar{\rho}}{\partial \eta}\right) - \frac{\partial u'/\bar{u}}{\partial \eta}\frac{d\bar{S}/C_P}{d\mu} = \omega\tau\frac{E_{y0}/B_0}{u_0}\left[2(\gamma - 1)\frac{jM^2}{\mu^2}\frac{d\mu\chi}{d\mu} + \gamma j\chi\frac{M^2}{\mu}\frac{d\bar{S}/C_P}{d\mu} + \frac{d}{d\mu}\left(\frac{\chi M^2}{\mu}j\right)\right] \quad (31)$$

Thus, a single equation in the single unknown $\partial(u'/\bar{u})/\partial\eta \equiv Y$ is finally obtained by subtracting the derivative with respect to μ of Eq. (30) from Eq. (31) to give

$$\mu\frac{\partial^2}{\partial \mu^2}(Y) + 3\frac{\partial}{\partial \mu}(Y) + \left(\frac{d\bar{S}/C_P}{d\mu}\right)Y = -\omega\tau\frac{E_{y0}/B_0}{u_0}\left[\frac{d}{d\mu}\left(j\frac{d\chi}{d\mu}\right) + 2(\gamma - 1)\frac{jM^2}{\mu^2}\frac{d\mu\chi}{d\mu} + \gamma\chi j\frac{M^2}{\mu}\frac{d\bar{S}/C_P}{d\mu} + \frac{d}{d\mu}\left(\frac{\chi M^2}{\mu}j\right)\right] \quad (32)$$

Equation (32) gives the perturbation in axial velocity in terms of the imposed channel-flow variables, and as is evident, the right sides of Eqs. (30–32) are functions of μ only.

Upon solution of this equation for Y, the group $\partial(\rho'/\bar{\rho})/\partial\eta$ is easily obtained from Eq. (30). Equation (21) may be modified to give an expression for the pressure, that is,

$$\frac{\partial P'/\bar{p}}{\partial \eta} = \omega\tau\frac{E_{y0}/B_0}{u_0}\gamma M^2\frac{j\chi}{\mu} \quad (33)$$

The perturbation temperature may then be obtained from the relationship

$$T'/\bar{T} = (P'/\bar{P}) - (\rho'/\bar{\rho}) \quad (34)$$

The perturbation in the x component of the electric field is imposed directly by the given channel-flow current field, and follows from the modified form of Eq. (24)

$$\frac{\partial E_x'}{\partial \eta} = \frac{d\mu}{d\xi}\frac{d\bar{E}_y}{d\mu} \quad (35)$$

Finally, Eq. (23) may be manipulated to give for the y component of the electric field

$$\frac{\partial}{\partial \eta}\left(\frac{E_y'}{E_{y0}}\right) = \frac{u_0}{E_{y0}/B_0}\left[\mu\chi Y - \omega\tau\frac{d\mu}{d\xi}\frac{d\mu\chi}{d\mu}\right] \quad (36)$$

Initial Conditions

Several papers (e.g., Refs. 5 and 6) have attempted to describe the effects of entry into electric and magnetic fields upon the mean fluid flow. All these solutions have involved perturbation techniques and, to the author's knowledge, have not included the effects of a Hall component of current. For this reason, a particularly simple form of initial conditions will be assumed here, but it should be realized that the form of solution of Eqs. (30–36) will not be affected by the choice of initial conditions. In fact, if desired, numerical solutions could be obtained to give the necessary property variations at a suitable station in the channel, and Eqs. (30–36) then used to give the variation of the fluid properties downstream. It will be seen that in most cases the effects of the initial conditions are soon damped out.

In this paper we shall assume that the fluid quickly enters a pressure field given by Eq. (33) at the inlet station. It is then assumed that no variation in axial velocity across the channel has occurred, and in addition no variation in entropy across the channel has occurred. This latter assumption will be justified if the "inlet region" is short, and it gives us the useful result that at entry the isentropic relationship between pressure and density holds. That is, to first order,

$$\frac{\partial}{\partial \eta}\left(\frac{P'}{\bar{P}}\right)\bigg|_0 = \gamma\frac{\partial}{\partial \eta}\left(\frac{\rho'}{\bar{\rho}}\right)\bigg|_0$$

With Eq. (33) we then find

$$\frac{\partial \rho'/\bar{\rho}}{\partial \eta}\bigg|_0 = \omega\tau\frac{E_{y0}/B_0}{u_0}j_0M_0^2 \quad (37)$$

With Eq. (30), together with the assumption of no axial velocity variation at entry, we then obtain for the initial conditions of the perturbation velocity

$$Y_0 = 0 \quad (38)$$

$$\frac{\partial Y}{\partial \mu}\bigg|_0 = -\omega\tau j_0\frac{E_{y0}/B_0}{u_0}\left[\frac{d\chi}{d\mu}\bigg|_0 + M_0^2\right] \quad (39)$$

Example Solutions

Several example solutions will now be worked, and for this purpose it will be convenient to refer to the results of the channel-flow analyses previously summarized in Table 1. With the results of Table 1 we see that Eq. (32) may be written in the following forms for each of the special cases considered in the Table:

Case 1

$$\mu^2 \frac{\partial^2}{\partial \mu^2}(Y) + 3\mu \frac{\partial}{\partial \mu}(Y) + \frac{\gamma + 1}{\gamma} Y = -\omega\tau \frac{(\gamma + 1)(2\gamma - 1)}{2\gamma(\gamma - 1)} \frac{1}{\mu^2} \quad (40)$$

Case 2

$$\mu^2 \frac{\partial^2}{\partial \mu^2}(Y) + 3\mu \frac{\partial}{\partial \mu}(Y) + \frac{\gamma + 1}{\gamma} Y = -\omega\tau \frac{(2\gamma + 1)(3\gamma - 1)}{2\gamma(\gamma - 1)} \quad (41)$$

Case 3

$$\mu^2 \frac{\partial^2}{\partial \mu^2}(Y) + 3\mu \frac{\partial}{\partial \mu}(Y) + \frac{\gamma - 1}{\gamma} Y = \frac{-\omega\tau}{(\gamma - 1)^2}\left[-\frac{\gamma(3 - \gamma)}{\mu^2} + \frac{7\gamma + 1}{\mu^4} - \frac{18}{\mu^6}\right] \quad (42)$$

Case 4

$$\mu^2 \frac{\partial^2}{\partial \mu^2}(Y) + 3\mu \frac{\partial}{\partial \mu}(Y) + \frac{\gamma + 1}{\gamma} \frac{\gamma - (\gamma - 1)\mu}{\gamma + 1 - (\gamma - 1)\mu} Y = 2\frac{\omega\tau}{\gamma - 1} \frac{\mu(\gamma - 1)(3\gamma - 1) - \gamma(3\gamma - 2)}{\gamma + 1 - (\gamma - 1)\mu} \quad (43)$$

Now, in cases 1–3 it can be seen that in each case the equation is equidimensional, and consequently may be integrated directly by elementary techniques. As could be expected beforehand, the influence of the heating, as reflected in the term $d\bar{S}/C_P/d\mu$ is critical in its effect on the equations, and the character of the solution depends greatly on the coefficients preceeding the term Y. Integration of Eqs. (40–42) yields:

Case 1

$$Y = -\omega\tau \frac{2\gamma - 1}{2(\gamma - 1)} \frac{1}{\mu^2} + \frac{1}{\mu}\left[\left\{Y_0 + \omega\tau \frac{2\gamma - 1}{2(\gamma - 1)}\right\} \cos(\ln \mu^{1/\gamma^{1/2}}) + \gamma^{1/2}\left\{\frac{\partial}{\partial \mu}(Y)|_0 - \omega\tau \frac{2\gamma - 1}{2(\gamma - 1)} + Y_0\right\} \sin(\ln \mu^{1/\gamma^{1/2}})\right] \quad (44)$$

Case 2

$$Y = -\omega\tau \frac{2\gamma + 1}{\gamma + 1} \frac{3\gamma - 1}{2(\gamma - 1)} + \frac{1}{\mu}\left[\left\{\omega\tau \frac{2\gamma + 1}{\gamma + 1} \frac{3\gamma - 1}{2(\gamma - 1)} + Y_0\right\} \cos(\ln \mu^{1/\gamma^{1/2}}) + \gamma^{1/2}\left\{\omega\tau \frac{2\gamma + 1}{\gamma + 1} \frac{3\gamma - 1}{2(\gamma - 1)} + Y_0 + \frac{\partial}{\partial \mu}(Y)|_0\right\} \sin(\ln \mu)^{1/\gamma^{1/2}}\right] \quad (45)$$

Case 3

$$Y = \frac{\omega\tau\gamma}{(\gamma - 1)^2}\left[\frac{\gamma(3 - \gamma)}{\gamma - 1} \frac{1}{\mu^2} - \frac{7\gamma + 1}{9\gamma - 1} \frac{1}{\mu^4} + \frac{18}{25\gamma - 1} \frac{1}{\mu^6}\right] + \frac{1}{\mu}\left[A\mu^{1/\gamma^{1/2}} + B\mu^{-1/\gamma^{1/2}}\right] \quad (46)$$

In this last equation, the constants A and B are given by:

$$A = \frac{\gamma^{1/2} + 1}{2} Y_0 + \frac{\gamma^{1/2}}{2} \frac{\partial}{\partial \mu}(Y)|_0 - \frac{\gamma^{1/2}}{2} \frac{\gamma}{(\gamma - 1)^2} \omega\tau \left[-\frac{\gamma(3 - \gamma)}{\gamma - 1} + \frac{3(7\gamma + 1)}{9\gamma - 1} - \frac{90}{25\gamma - 1}\right] - \frac{\gamma}{2} \frac{\omega\tau}{(\gamma - 1)^2}\left[\frac{\gamma(3 - \gamma)}{\gamma - 1} - \frac{7\gamma + 1}{9\gamma - 1} + \frac{18}{25\gamma - 1}\right]$$

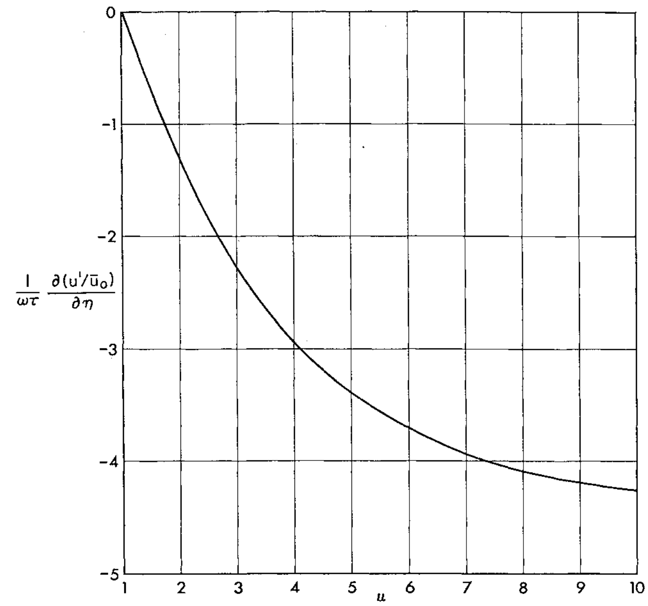

Fig. 5 Graph of slope of perturbation velocity $[\partial(u'/\bar{u}_0)/\partial\eta]$ vs dimensionless channel-flow velocity ($\gamma = \frac{5}{3}$, case 1).

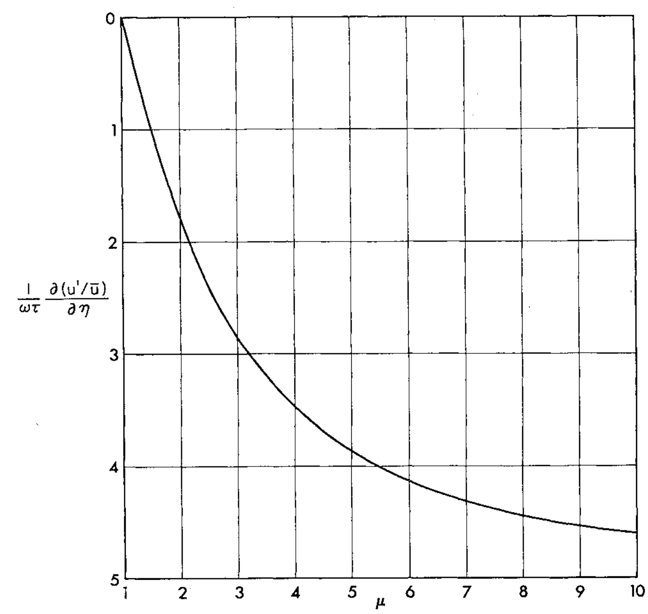

Fig. 6 Graph of slope of perturbation velocity $[\partial(u'/\bar{u})/\partial\eta]$ vs dimensionless channel-flow velocity ($\gamma = \frac{5}{3}$, case 2).

$$B = -\frac{\gamma^{1/2} - 1}{2} Y_0 - \frac{\gamma^{1/2}}{2} \frac{\partial}{\partial \mu}(Y)|_0 + \frac{\gamma^{1/2}}{2} \frac{\gamma}{(\gamma - 1)^2} \omega\tau \left[-\frac{\gamma(3 - \gamma)}{\gamma - 1} + \frac{3(7\gamma + 1)}{9\gamma - 1} - \frac{90}{25\gamma - 1}\right] - \frac{\gamma}{2} \frac{\omega\tau}{(\gamma - 1)^2}\left[\frac{\gamma(3 - \gamma)}{\gamma - 1} - \frac{7\gamma + 1}{9\gamma - 1} + \frac{18}{25\gamma - 1}\right]$$

These rather unwieldy solutions are given in terms of the general inlet conditions Y_0 and $(\partial/\partial\mu)(Y)|_0$. Using the special inlet conditions given by Eqs. (38) and (39) these re-

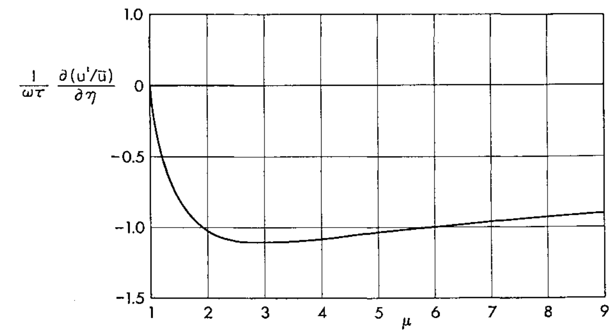

Fig. 7 Graph of slope of perturbation velocity $[\partial(u'/\bar{u})/\partial\eta]$ vs dimensionless channel-flow velocity ($\gamma = \frac{5}{3}$, case 3).

sults may be simplified considerably. Thus, with the results of Table 1, and rearranging the oscillating terms so as to clearly indicate the amplitude of the oscillations, the following results are obtained:

Case 1

$$\mu Y = \omega\tau \frac{2\gamma - 1}{2(\gamma - 1)} \left\{ \left(\frac{\gamma + 1}{\gamma} \frac{4\gamma^3 + 1}{(2\gamma - 1)^2} \right)^{1/2} \times \cos\left[\ln(\mu^{1/\gamma^{1/2}}) + \operatorname{arc\,cos} \left(\frac{\gamma(2\gamma - 1)^2}{(\gamma + 1)(4\gamma^3 + 1)} \right)^{1/2} \right] - \frac{1}{\mu} \right\} \qquad (47)$$

Case 2

$$Y = \omega\tau \frac{2\gamma + 1}{2(\gamma + 1)} \frac{3\gamma - 1}{\gamma - 1} \times \left\{ -1 + \frac{1}{\mu} \left(1 + \frac{(\gamma - 1)^2(3\gamma + 1)^2}{\gamma(3\gamma - 1)^2} \right)^{1/2} \times \cos\left[\ln(\mu^{1/\gamma^{1/2}}) - \operatorname{arc\,cos} \frac{1}{\left(1 + \frac{(\gamma - 1)^2(3\gamma + 1)^2}{\gamma(3\gamma - 1)^2} \right)^{1/2}} \right] \right\} \qquad (48)$$

For case 3 the equation remains the same as Eq. (46), but the constants reduce to

$$A = -\omega\tau \frac{\gamma^{1/2}}{2(\gamma - 1)^2} \left[2 + \gamma \left\{ -\frac{\gamma(3 - \gamma)}{\gamma - 1} + \frac{3(7\gamma + 1)}{9\gamma - 1} - \frac{90}{25\gamma - 1} \right\} + \gamma^{1/2} \left\{ \frac{\gamma(3 - \gamma)}{\gamma - 1} - \frac{7\gamma + 1}{9\gamma - 1} + \frac{18}{25\gamma - 1} \right\} \right]$$

$$B = \omega\tau \frac{\gamma^{1/2}}{2(\gamma - 1)^2} \left[2 + \gamma \left\{ -\frac{\gamma(3 - \gamma)}{\gamma - 1} + \frac{3(7\gamma + 1)}{9\gamma - 1} - \frac{90}{25\gamma - 1} \right\} - \gamma^{1/2} \left\{ \frac{\gamma(3 - \gamma)}{\gamma - 1} - \frac{7\gamma + 1}{9\gamma - 1} + \frac{18}{25\gamma - 1} \right\} \right]$$

Graphical results from these equations for a ratio of specific heats γ of $\frac{5}{3}$ are contained in Figs. 5 to 7. The results for case 4 are shown in Fig. 8. These latter results were obtained by computer. Further results may be easily obtained from Eqs. (33–36).

Interpretation of the Results

Inspection of Eq. (47) shows an interesting feature exhibited by the imposed channel-flow conditions of case 1. That is, if one considers the asymptotic behavior of the equation for large channel-flow velocities, it can be seen that the slope of the dimensionless velocity perturbation $\mu Y = \partial(u'/\bar{u}_0)/\partial\eta$ oscillates about zero with a constant amplitude. This somewhat surprising limiting behavior can be explained when the imposed channel-flow conditions are considered. Thus, the fact that the solution imposes constant current throughout leads to the condition of constant body force and constant ohmic heating across the channel. Initially, then, the greater mass flow per unit area experienced in the upper portion of the channel (introduced by the induced pressure field causing a density increase) brings about a reduction in velocity compared to that in the lower portion of the channel. Eventually, the mass flow per unit area will become uniform across the channel, at which time from the point of view of the momentum equation, acceleration would be uniform across the channel.

The energy and state equations, however, then reveal that, at this point, the uniform heating per unit volume experienced across the channel will introduce larger negative density changes at the top of the channel because of the larger density existing there. That is, one may write from the equation of state,

$$\Delta\rho/\rho = (\Delta P/P) - (\Delta T/T)$$

The channel flow conditions of case 1 indicate that the temperature variation is more rapid than that of the pressure (i.e., $P \alpha \bar{u}$ while $T \alpha \bar{u}^2$), so that the second term of this equation becomes dominant. Then one has approximately,

$$\Delta\rho \ \alpha \ - \rho^2 \Delta T$$

Thus, as just stated, the uniform heating, (uniform ΔT) at this point of uniform mass-flow per unit area, introduces a larger (negative) change in density at the top of the channel than at the bottom. This in turn leads to a reduced mass-flow per unit area in the region that previously had an excess mass-flow per unit area. Eventually the Lorentz forces will accelerate the fluid to such an extent that the mass-flow per unit area will be uniform across the channel again, but at this time the situation will be reversed and the top of the channel will now have an excess of velocity and the bottom an excess of density. The entire procedure then tends to reverse itself leading to the oscillatory motion given by Eq. (47).

For case 2, it is seen that the continually increasing energy input per unit volume, found downstream, soon dominates any entrance conditions, and the slope of the velocity perturbation Y, asymptotically approaches the value

$$-\omega\tau \frac{2\gamma + 1}{2(\gamma + 1)} \frac{3\gamma - 1}{\gamma - 1}$$

For a noble gas this asymptotic value is $-\frac{39}{8}\omega\tau$. It is then apparent that the linearized theory, and particularly the modified channel-flow theory, will be valid only for very narrow

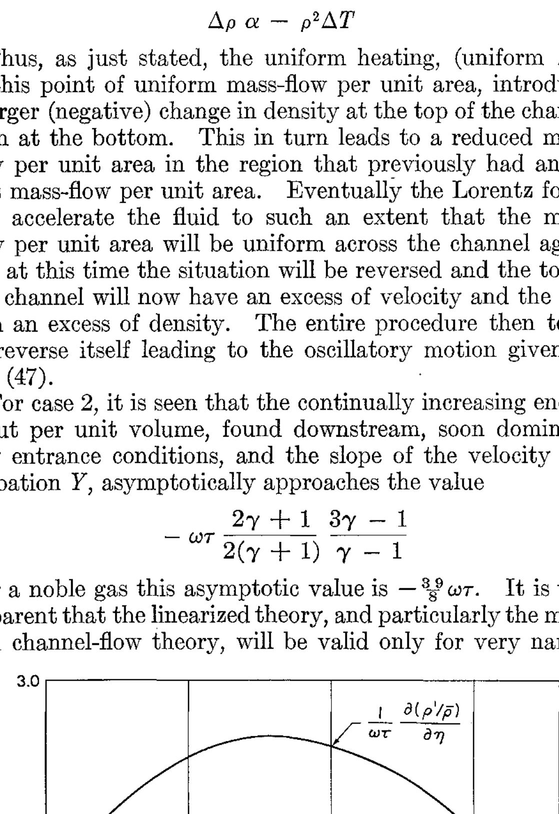

Fig. 8 Graph of slope of dimensionless perturbations in mass flow $\{[\partial(\rho'/\bar{\rho})/\partial\eta] + [\partial(u'/\bar{u})/\partial\eta]\}$, velocity $[\partial(u'/\bar{u})/\partial\eta]$, and density $[\partial(\rho'/\bar{\rho})/\partial\eta]$ vs dimensionless channel-flow velocity ($\gamma = \frac{5}{3}$, case 4).

channels. That is, we have

$$\frac{u'}{\bar{u}} = -\frac{\omega\tau}{1+(\omega\tau)^2}\frac{39}{16}\frac{H}{\bar{\rho}\bar{u}/\sigma B_0^2} = -\frac{39}{32}\frac{H}{\bar{\rho}\bar{u}/\sigma B_0^2} \text{ for } \omega\tau = 1$$

Thus, the dimensionless length $H/(\bar{\rho}\bar{u}/\sigma B_0^2)$ must be kept of low order for the analysis to be valid. For a highly loaded machine, this could conceivably demand a channel so narrow that viscous effects could invalidate the assumptions of the channel-flow theory.

The results of case 3 and case 4 tend to be intermediate in behavior between the cases just discussed. It should be noted that in case 3 the perturbation expressed as $u'/\bar{u}$ decreases with increasing channel-flow velocity, but when expressed as $u'/\bar{u}_0$ the perturbation increases with increasing channel-flow velocity. The results of case 4 will of course be in doubt for the higher values of the channel-flow velocity, because the great axial length of machine required to produce the higher channel-flow velocities will introduce sizeable viscous effects.

Application of Theory to Channels with Asymmetric Inlet Conditions but no Axial Component of Current

The theory developed in the preceeding sections can be applied, with little modification, to the case of a fluid with asymmetric inlet conditions flowing into a channel with segmented electrodes. In this case, no axial current will flow, so that no pressure gradient in the y direction will exist. The effect on Eqs. (40–43) is simply to cause the inhomogeneous side of the equations to vanish (in effect, to cause the Hall parameter to become zero).

The solutions to the equations, as given by Eqs. (44–46) are thus valid, except that one must put $\omega\tau = 0$.

The results are then

Cases 1 and 2

$$\mu Y = Y_0 \cos(\ln\mu^{1/\gamma^{1/2}}) + \gamma^{1/2}\left\{\frac{\partial}{\partial\mu}(Y)|_0 + Y_0\right\}\sin(\ln\mu^{1/\gamma^{1/2}}) \tag{49}$$

Case 3

$$\mu Y = \left[\left\{\frac{\gamma^{1/2}+1}{2}Y_0 + \frac{\gamma^{1/2}}{2}\frac{\partial}{\partial\mu}(Y)|_0\right\}\mu^{1/\gamma^{1/2}} - \left\{\frac{\gamma^{1/2}-1}{2}Y_0 + \frac{\gamma^{1/2}}{2}\frac{\partial}{\partial\mu}(Y)|_0\right\}\mu^{-1/\gamma^{1/2}}\right] \tag{50}$$

Initial Conditions

Equation (30) allows us to obtain $(\partial/\partial\mu)(Y)|_0$ in terms of the inlet density perturbations, from which we find,

$$\frac{\partial}{\partial\mu}(Y)\bigg|_0 = -\frac{\partial\rho'/\bar{\rho}}{\partial\eta}\bigg|_0 - 2Y_0 \tag{51}$$

This expression may then be substituted into Eqs. (49) and (50) to give the velocity perturbations as a function of the channel velocity. Our greatest interest lies in the investigation of the growth or decay of inlet perturbations, so for simplicity let us consider the situation where there is a velocity perturbation at inlet, but no density perturbation. Then, with Eq. (51) substituted into Eqs. (49) and (50) we find

Cases 1 and 2

$$\mu Y = (\gamma+1)^{1/2}Y_0\cos\left\{\ln\mu^{1/\gamma^{1/2}} + \arccos\frac{1}{(\gamma+1)^{1/2}}\right\} \tag{52}$$

Case 3

$$\mu Y = Y_0\left[-\frac{\gamma^{1/2}-1}{2}\mu^{1/\gamma^{1/2}} + \frac{\gamma^{1/2}+1}{2}\mu^{-1/\gamma^{1/2}}\right] \tag{53}$$

These results are most interesting, in that one sees that in all cases, the perturbation of the inlet velocity in terms of the inlet channel velocity, $u'/\bar{u}_0$, grows as one proceeds downstream.

Thus, for cases 1 and 2 we see that magnitude of the perturbation is multiplied by a maximum factor of $(\gamma + 1)^{1/2}$. In case 3 a perturbation in $u'/\bar{u}_0$ will grow slowly, but indefinitely, downstream. The perturbation as a fraction of the local channel-flow velocity $u'/\bar{u}$, however, decreases downstream. Any density perturbation could be obtained by substituting the results of Eqs. (49) and (50) into Eq. (30) (the latter with $\omega\tau = 0$).

Conclusions

The results obtained from the theory developed in this paper indicate that for a Hall parameter of the order of unity or greater, perturbations in the axial velocity and thermodynamic properties of the fluid will be induced, and will persist throughout the accelerator length. The quantitative effects vary for the various prescribed field loadings of course, but it has been shown that the ratio $H/(\bar{\rho}\bar{u}/\sigma B_0^2)$ must be kept much less than unity if the channel-flow solutions are to remain valid. The first-order effects of finite $H/(\bar{\rho}\bar{u}/\sigma B_0^2)$, are given by the theory.

One thus sees that in machines with high electro-magnetic loading (i.e., small $\bar{\rho}\bar{u}/\sigma B_0^2$), the channel-flow solutions would be of limited validity. That is, if large fluid acceleration were obtained, the ratio $L/(\bar{\rho}\bar{u}/\sigma B_0^2)$ would be large, and the group $H/L[L/(\rho\bar{u}/\sigma B_0^2)]$ would be small only if the geometry of the accelerator were such that H/L was very small. If this were the case, viscous effects would become important, and again, the channel-flow results would be questionable.

Finally, the investigation into the behavior of a perturbation in a fluid property found at entrance to an accelerator in which no axial current exists, showed that the absolute value of such a perturbation could grow downstream. In all cases investigated, however, the fractional perturbation $u'/\bar{u}$ decreased downstream.

The equations developed in the report [in particular Eqs. (30–36)] are in a form such that the results of any given channel-flow analysis may be easily substituted to allow solution for the various perturbation quantities.

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